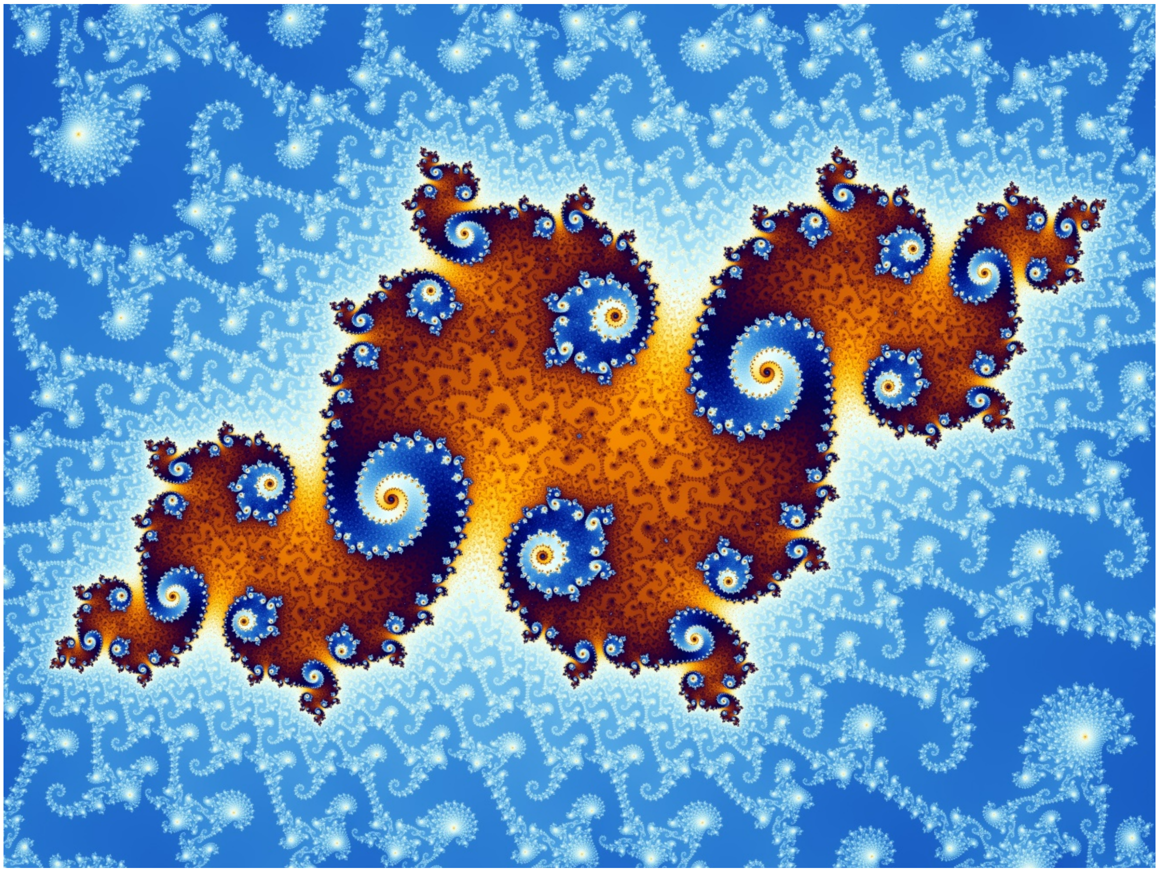


Fractals

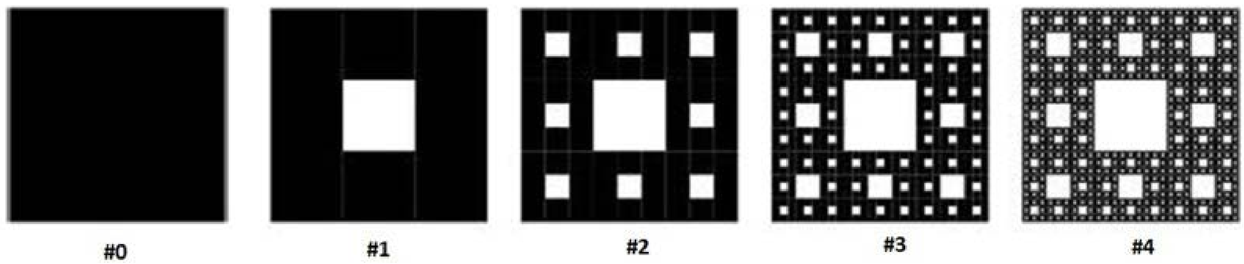
> The existence of these patterns [fractals] challenges us to study forms that Euclid leaves aside as being formless, to investigate the morphology of the amorphous. Mathematicians have disdained this challenge, however, and have increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel.
>
> **Benoit Mandelbrot** (Polish/French mathematician; 1924 - 2010)

One way to build self-similar fractals is to use a *iterative process* where the shape is built on smaller and smaller scales by the application of a single set of defining rules:[1]

- Choose an **Initiator** which is your starting shape.
- Choose a **Generator** which is a collection of scaled copies of the Initiator.
- In each stage replace every copy of the Initiator in the existing stage with an appropriately scaled copy of the Generator.
- Continue this process infinitely, arriving at a self-similar fractal in the *limit*.

[1] The approach and terminology used here are from Michael Frame's Fractal course taught for many years at Yale University.

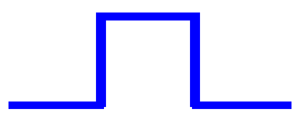
When naming the stages, the initiator is generally considered stage 0 and the generator stage 1. The fractal below is known as a Sierpinski Carpet. And yes, if you create a new unknown fractal, you can name it after yourself!



1. Describe how many scaled copies of the Generator (stage 1) are found in stage 2.

Refer to the initiator and generator below, to create your very own fractal.

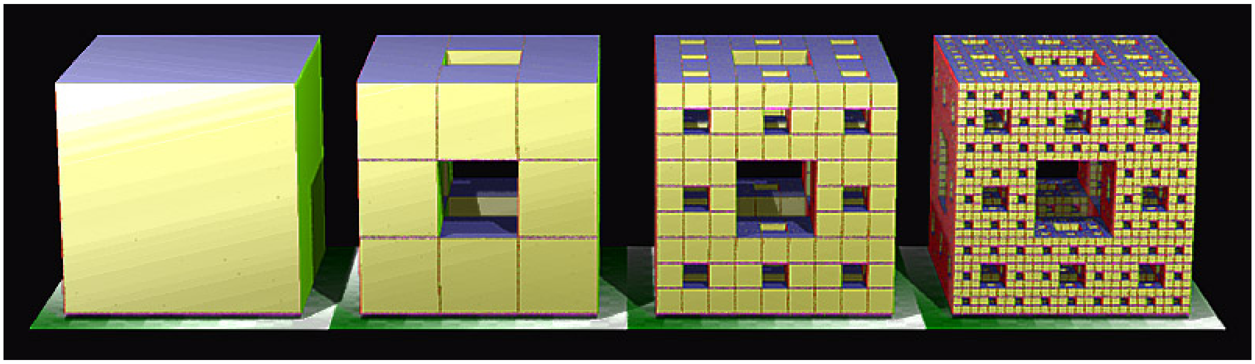
Initiator:

Generator: 

Hint: Stage 0 and Stage 1 are just copied exactly. To create stage 2, simply replace each straight segment in the generator with a scaled down copy of itself.

2. Draw stage 2 of the fractal below.

3. Draw stage 3 of the fractal below.



Pictured above is the **Menger Sponge** invented by **Karl Menger** (Austrian-American mathematician; 1902 - 1985) in 1926. The initiator is simply a unit cube, shown on the left. The generator is the figure second from the left. One way to think about the generator is that its is formed by cross-sectioning the initiator by cuts that are parallel to each face and trisect the edges, then removing all cubies on each face as well as the one in the center.
Another way to think of the generator is through the *iterative* process that defines the fractal in the limit.

4. How many copies of the Initiator are used to create the Generator?

5. How are the linear dimensions scaled as the initiator is used in creating the generator? (This is called the *linear scaling factor*.)

6. What is the volume of each of the scaled copies of the Initiator in the Generator?

7. What is the volume of the Generator? (Hint: Here and below it will help in locating patterns if you leave your results as fractions.)

8. How many copies of the Generator are used in creating Stage 2?

9. What is the volume of each of the scaled copies of the Generator in Stage 2?

10. What is the volume of Stage 2?

11. How many copies of Stage 2 are used in creating Stage 3?

12. What is the volume of each of the scaled copies of Stage 2 in Stage 3?

13. What is the volume of Stage 3?

14. Find an expression for the volume of Stage *n*.

15. Make a table and/or plot data for values for *n* to analyze what happens to the volume in Stage *n* for larger and larger values of *n.*

The Menger Sponge is the limiting object obtained from this process. Although one starts with a cube, the process pokes an unbounded number of (square) holes through the cube. So many (square) holes are poked that the Menger Sponge has zero volume (as results in Investigation **15** should show)! There are so many holes that if feels almost as if the Menger Sponge becomes invisible. Has it slipped out of the realm of being three-dimensional?

A wonderful thing about natural and synthetic sponges is their ability to "soak up" water. How do they work? They have an excessive number of pores throughout their volume. These pores contribute enormous surface area relative to their volume and water has all of this surface area to adhere to.

If we could actually make one, the Menger Sponge would be the ultimate sponge, for it has infinite surface area! (See the Further Investigations Section below for details.)

A finite object with zero volume and yet infinite surface area? What dimension does this remarkable object naturally occupy?

16. Use a separate piece of paper to create at least ten iterations of a Pythagorean Tree Fractal. Refer to the initiator and generator below: